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Mean magnetic field and noise cross-correlation in magnetohydrodynamic turbulence: results from a one-dimensional model

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Abstract. We show that a recently proposed [J. Fleischer, P.H. Diamond, Phys. Rev. E **58**, R2709 (1998)] one-dimensional Burgers-like model for magnetohydrodynamics (MHD) is in effect identical to existing models for drifting lines and sedimenting lattices. We use the model to demonstrate, contrary to claims in the literature, that the energy spectrum of MHD turbulence should be independent of mean magnetic field and that cross-correlations between the noise sources for the velocity and magnetic fields cannot change the structure of the equations under renormalisation. We comment on the scaling and the multiscaling properties of the stochastically forced version of the model.

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1 Introduction and results

The search for simple model equations embodying some of the features of fluid turbulence prompted Burgers to propose his famous nonlinear diffusion equation [1]

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u = \nu_0 \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad (1)$$

in one space dimension, where ν_0 is a "viscosity" and f a forcing function. The unique properties of (1), particularly the Cole-Hopf transformation [1] that linearises it, together with its obvious formal similarity to the Navier-Stokes equation, have led to many studies [2,3] shedding some light on real fluid turbulence. Turbulence in *plasmas* is believed to be described by the equations of magnetohydrodynamics [4] in three dimensions (3dMHD) for the coupled evolution of the velocity field $\mathbf{u}$ and the magnetic field $\mathbf{B}$. In 3dMHD, the Navier-Stokes equation for an incompressible fluid is modified by the inclusion of electromagnetic stresses:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi\rho} + \nu\nabla^2\mathbf{u} + \mathbf{f}_u \qquad (2)$$

with $\nabla\cdot\mathbf{u} = \mathbf{0}$; and Ampère's law for a conducting fluid becomes

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \mu\nabla^2\mathbf{B} + \mathbf{f}_b. \qquad (3)$$

In (2) and (3), ρ is the fluid density, p is the pressure, μ is the "magnetic viscosity", arising from the nonzero resistivity of the plasma, ν is the kinematic viscosity and $\mathbf{f}_u$ and $\mathbf{f}_b$ are forcing functions. Since the MHD equations are distinctly more complicated than the Navier-Stokes equation, a toy model, whose relation to the MHD equations is the same as that of the Burgers equation to the Navier-Stokes equation, should be very welcome.

The first attempt in this direction [5] lacked cross-helicity conservation. The 1d model with all the *scalar* conservation laws of 3dMHD is that due to Fleischer and Diamond [6]. Their equations, however, have been in the literature for some time now in a different context. For the case where the mean magnetic field $b_0 = 0$, the equations are those of the Ertaş-Kardar [7] model of drifting lines. For the general case $b_0 \neq 0$, they are the coupled equations, for the displacements along and normal to the drift direction, of the one-dimensional reduced model of Lahiri and Ramaswamy [8] for a crystalline lattice moving through a dissipative medium. This surprising relation arises from an exact correspondence between the symmetries of these equations and those of the 1dMHD equations [6] with and without $b_0 = 0$. The relation between the velocity and magnetic fields in [6] and the displacement fields in [7,8] is precisely the same as that between the height in the KPZ equation [9] and the velocity in (1).

Reduced models are useful if they allow one to answer a question of principle. We use the 1dMHD model to show that the form of the energy spectrum in MHD with a mean magnetic field should be the same as that

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without one, contrary to a claim by Kraichnan [10]. We also explore the scaling and multiscaling properties of the model equations in the presence of stochastic forcing. For a temporally white noise source with variance $\propto k^{-y+3}$ at small wavenumbers k, we find "sweeping" divergences for $y \geq 3$ and, within perturbative and renormalisation-group analyses, we show that the effective kinematic and magnetic viscosities are equal at long wavelengths. Specifically for $y = 4$, as is appropriate [2,3] for a study of 1d turbulence, we find within a scaling treatment that the energy spectrum $E(k) \sim k^{-5/3}$ and the dynamic exponent $z = 2/3$. We show also that the multiscaling properties of the model should be precisely the same as that of a Burgers equation with the same driving [3]. Lastly, we correct a claim [11] in the literature regarding the role of cross-correlations between the noise sources in the velocity and magnetic-field equations.

2 Symmetries, equations of motion, and noise statistics

Let us remind the reader of the "derivation" of the 1dMHD equations [6] emphasising some points which we feel are important and are missing in reference [6]. The structure of the 1d model is determined by the invariances of the MHD equations. (i) The velocity and magnetic fields are polar and axial vectors respectively. For a model in one space dimension with one-component fields $u(x)$ for the velocity and $b(x)$ for the magnetic field this implies evolution equations which are invariant under $x \to -x$ together with $u \to -u$, with $b \to b$. (ii) The MHD equations are Galileian-invariant, so our 1d model must be unchanged under $x = x' + u_0 t'$, $t = t'$, $u(x,t) = u'(x',t') + u_0$, $b(x,t) = b'(x',t')$ for any constant u_0. (iii) The 1d equations, like the 3d originals, must have the "conservation law" form $\partial_t(\ ) = \nabla.(\ )$. (iv) The equations, in the absence of the diffusion terms, must conserve the "energy"

$$\mathcal{E} \equiv \int (u^2 + \beta b^2) \mathrm{d}x \tag{4}$$

for some constant β. To leading orders in gradients and bilinear order in fields, the most general 1dMHD model consistent with these requirements [6] is [1]

$$\frac{\partial u}{\partial t} + \lambda_2 b_0 \frac{\partial b}{\partial x} + \lambda_1 u \frac{\partial u}{\partial x} + \lambda_2 b \frac{\partial b}{\partial x} = \nu_0 \frac{\partial^2 u}{\partial x^2} + f_u, \tag{5a}$$

$$\frac{\partial b}{\partial t} + \lambda_1 b_0 \frac{\partial}{\partial x} u + \lambda_1 \frac{\partial}{\partial x}(ub) = \mu_0 \frac{\partial^2 b}{\partial x^2} + f_b. \tag{5b}$$

Integrating (5) once with respect to x, one finds that $\int_x u$ and $\int_x b$ obey the equations of Ertaş and Kardar [7] for drifting lines when $b_0 = 0$, and those of Lahiri and Ramaswamy [8] for sedimenting lattices when $b_0 \neq 0$. This transformation of variables is the same as that which takes one from the Burgers equation to the KPZ [9] equation.

In (5), the constant b_0 is the "mean magnetic field", ν and μ are the kinematic and magnetic viscosities, and f_u and f_b are forcing functions. λ_1 can always be rescaled to unity but is retained for book-keeping purposes. λ_2 is arbitrary, and all statements made in this paper hold independent of its value [6]. It is straightforward to show that the equations, in the absence of the diffusion terms, conserve the energy-like quantity $\mathcal{E}$ defined in equation (4), for $\beta = \lambda_2/\lambda_1$, as well as $\int_x ub$, the analogue in $d = 1$ of the cross helicity ($\int_x \mathbf{u} \cdot \mathbf{B}$) [4], and that the model of [5] does not conserve the latter. A third conserved quantity, the magnetic helicity $\int \mathbf{A} \cdot \mathbf{B}$, where $\mathbf{A}$ is the vector potential, has no analogue in $d = 1$. It is important to note that for $b_0 = 0$ the equations have a higher symmetry than for $b_0 \neq 0$. For both $b_0 = 0$ and $b_0 \neq 0$, they are of course invariant under $x \to -x, u \to -u, b \to b$, as imposed by the transformation properties of the fields. For $b_0 = 0$, they are *in addition* invariant under $x \to -x, u \to -u, b \to -b$. For $\lambda_1 \lambda_2 > 0$, in the linear approximation at small wavenumber k, as discussed in [8], the equations (5) have a wavelike response with frequency $\omega = \pm\sqrt{\lambda_1 \lambda_2 b_0^2}$. These are the 1d analogue of Alfvén waves [4,6]. The linearly unstable case $\lambda_1 \lambda_2 < 0$ [8] will not be considered here since the corresponding possibility is unphysical in the context of the 3dMHD equations (2) and (3).

Since we wish to study turbulence in (5), we shall take the forcing terms f_u, f_b to be zero-mean Gaussian noise sources with covariance specified below. The properties of u and b under $x \to -x$ and the reality of $f_u(x,t)$ and $f_b(x,t)$ imply that $\langle f_u(k,0) f_u(-k,t) \rangle$ and $\langle f_b(k,0) f_b(-k,t) \rangle$ are real and even in k while, crucially, $\langle f_u(k,0) f_b(-k,t) \rangle$ (if nonzero) is odd in k and purely imaginary. In general, therefore,

$$\langle f_u(k,0) f_u(k',t) \rangle = A_u(|k|) \delta(k+k') \delta(t) \tag{6a}$$

$$\langle f_b(k,0) f_b(k',t) \rangle = A_b(|k|) \delta(k+k') \delta(t) \tag{6b}$$

$$\langle f_u(k,0) f_b(-k,t) \rangle = ikC(|k|) \delta(k+k') \delta(t), \tag{6c}$$

where A_u, A_b and C are *real* functions. By strict analogy with the random stirring approach in $d = 3$ [12,13] and 1d Burgers turbulence [3] we choose

$$A_u(k) = \frac{\epsilon_u}{|k|} \tag{7}$$

so that the noise strength ϵ_u has the units of the dissipation parameter, *i.e.* (length)^2/(time)^3. $\langle f_u f_b \rangle$ and $\langle f_b f_b \rangle$ can be taken to be zero or at any rate no more singular than $\langle f_u f_u \rangle$ without altering any of our main conclusions. A self-consistent treatment starting with only a nonzero $\langle f_u f_u \rangle$ will generate $\langle f_u f_b \rangle$ and $\langle f_b f_b \rangle$ as well. In that sense, contrary to the statements in [6], the case where only one of the equations has a bare noise is not physically distinct from that where both noises are non-zero.

[1] The choice of bookkeeping coefficients in the equations as presented in [6] seems unnatural to us. Galileian invariance requires equality of the coefficients of $u\partial_x u$ and $\partial_x(ub)$, while that of $b\partial_x b$ can be left entirely arbitrary.

For convenience in carrying out bare perturbation theory, however, we take all components of the noise covariance to be as singular as $\langle f_u f_u \rangle$:

$$A_b(|k|) = \frac{\epsilon_b}{|k|} \tag{8}$$

$$kC(|k|) = \frac{\epsilon_{ub}}{k} \cdot \tag{9}$$

Note that it is $1/k$ and not $1/|k|$ that appears on the right-hand side of (9). We now use the reduced equations (5) to obtain the results advertised in the introduction.

3 Energy spectrum and the role of a mean magnetic field

Does the presence of a mean magnetic field b_0 affect the energy spectrum $E(k)$ at wavenumber k in MHD turbulence? It has been argued [10] that if $b_0 \neq 0$ then (i) the dissipation ϵ must be proportional to the Alfvén wave propagation time $1/ck$, where c is the wavespeed; (ii) ϵ must depend, apart from this, only on k and $E(k)$; and (iii) as a result of energy conservation and a local-cascade picture, ϵ must be independent of k. Taken together, these imply [10] that $E(k) \sim k^{-3/2}$, while for $b_0 = 0$ the usual Kolmogorov arguments yield $k^{-5/3}$.

The questionable assumption in the foregoing analysis is that the wave-propagation time sets the scale for the dynamics. In what follows, we test that assumption in the 1dMHD (5) model since the above arguments, if correct, should apply there as well. We show below explicitly that the presence of a nonzero wavespeed leaves the equal-time correlations unchanged at small k. We begin by rescaling $\lambda_1 u \to u$ and rewriting (5) in terms of the Elsässer [14] variables

$$z^{\pm} = \frac{u \pm \sqrt{\lambda_2} b}{\sqrt{1+\lambda_2}} \cdot \tag{10}$$

As has already been noted in [7,8], the equations for z^+ and z^- decouple completely [6] at small wavenumbers [2]:

$$\frac{\partial z^+}{\partial t} - \sqrt{\lambda_2} b_0 \partial_x z^+ + \frac{1}{2}\frac{\sqrt{1+\lambda_2}}{2}\partial_x {z^+}^2 = D\partial_{xx} z^+ + \ldots + f^+; \tag{11a}$$

$$\frac{\partial z^-}{\partial t} + \sqrt{\lambda_2} b_0 \partial_x z^- + \frac{1}{2}\frac{\sqrt{1+\lambda_2}}{2}\partial_x {z^-}^2 = D\partial_{xx} z^- + \ldots + f^-, \tag{11b}$$

with

$$f^{\pm} = \frac{f_u \pm \sqrt{\lambda_2} f_b}{\sqrt{1+\lambda_2}}, \tag{12}$$

[2] The decoupling is demonstrated in [6] for zero mean magnetic field, with equal magnetic and kinematic viscosities. Our treatment is somewhat more general.

where $D = (\mu + \nu)/2$, and the ellipsis refers to reactive (*i.e.*, nondissipative) terms proportional to $\nu - \mu$, subdominant in wavenumber relative to the leading Alfvén wave terms $\propto b_0$. It therefore suffices to study the case $\nu = \mu = D$. In (11a) and (11b), the wave terms $\propto b_0$ can be absorbed by opposite Galileian boosts, $x \to x - \sqrt{\lambda_2} b_0 t$, and $x \to x + \sqrt{\lambda_2} b_0 t$ respectively. Each equation can therefore be studied independently, at small wavenumber, as far as the autocorrelations of z^+ or z^- are concerned. However, the correlations of the physical fields u and b involve correlations $\langle z^+ z^- \rangle$ (which are nonzero because $\langle f^+ f^- \rangle \neq 0$). Some care must be taken while evaluating these, as we will show in detail below. Since

$$u = \frac{\sqrt{1+\lambda_2}}{2}(z^+ + z^-) \tag{13}$$

and

$$b = \frac{\sqrt{1+\lambda_2}}{2\sqrt{\lambda_2}}(z^+ - z^-) \tag{14}$$

we see that

$$\begin{aligned}\langle u(k,t)u(-k,0)\rangle = \frac{1+\lambda_2}{4}&[\langle z^+(k,t)z^+(-k,t)\rangle \\ &+ \langle z^-(k,t)z^-(-k,t)\rangle \\ &+ 2\mathrm{Re}(\langle z^+(k,t)z^-(-k,0)\rangle)],\end{aligned} \tag{15}$$

$$\begin{aligned}\langle b(k,t)b(-k,0)\rangle = \frac{1+\lambda_2}{4\lambda_2}&[\langle z^+(k,t)z^+(-k,t)\rangle \\ &+ \langle z^-(k,t)z^-(-k,t)\rangle \\ &- 2\mathrm{Re}(\langle z^+(k,t)z^-(-k,0)\rangle)]\end{aligned} \tag{16}$$

and

$$\begin{aligned}\langle u(k,t)b(-k,0)\rangle = \frac{1+\lambda_2}{4\sqrt{\lambda_2}}&[\langle z^+(k,t)z^+(-k,t)\rangle \\ &- \langle z^-(k,t)z^-(-k,t)\rangle \\ &- 2\mathrm{Im}(\langle z^+(k,t)z^-(-k,0)\rangle)].\end{aligned} \tag{17}$$

Let us define shifted coordinates

$$y_{\pm} = x \mp \sqrt{\lambda_2} b_0 t \tag{18}$$

and "comoving" fields

$$\zeta^{\pm}(y_{\pm},t) \equiv \zeta^{\pm}(x \mp \sqrt{\lambda_2} b_0 t, t) = z^{\pm}(x,t). \tag{19}$$

Then, Fourier transforming and defining

$$c = \sqrt{\lambda_2} b_0, \tag{20}$$

we get

$$z^{\pm}(k,t) = \mathrm{e}^{\mp ikct}\zeta^{\pm}. \tag{21}$$

The autocorrelations of z^+ and ζ^+, as well as those of z^- and ζ^-, are related by travelling waves:

$$\langle z^{\pm}(k,t)z^{\pm}(k',t')\rangle = \mathrm{e}^{\mp ikc(t-t')}\langle \zeta^{\pm}(k,t)\zeta^{\pm}(k',t')\rangle, \tag{22}$$

while that between z^+ and z^- has a phase factor which depends on the *sum* of the time arguments [3]:

$$\langle z^+(k,t) z^-(k',t')\rangle = \mathrm{e}^{-ikc(t+t')}\langle \zeta^+(k,t)\zeta^-(k',t')\rangle. \quad (23)$$

Defining

$$\tilde{\lambda} \equiv \frac{\sqrt{1+\lambda_2}}{2}, \quad (24)$$

we see that the fields $\zeta^\pm$ obey the Burgers equation

$$\partial_t \zeta_\pm + \frac{\tilde{\lambda}}{2}\partial_y \zeta_\pm^2 = D\partial_{yy}\zeta_\pm + \phi_\pm(y,t) \quad (25)$$

with noise correlations related to those of f^+ and f^- in equation (12) by

$$\langle \phi^+(k,t)\phi^+(k',t')\rangle = \langle f^+(k,t) f^+(k',t')\rangle, \quad (26)$$

$$\langle \phi^-(k,t)\phi^-(k',t')\rangle = \langle f^-(k,t) f^-(k',t')\rangle, \quad (27)$$

and

$$\langle \phi^+(k,t)\phi^-(k',t')\rangle = \mathrm{e}^{ikc(t+t')}\langle f^+(k,t) f^-(k',t')\rangle. \quad (28)$$

Despite the appearance of nonstationary phase factors in the correlations $\langle\phi^+\phi^-\rangle$ and $\langle\zeta^+\zeta^-\rangle$, as a result of the time-dependent coordinate transformation, all physical correlation functions will of course be time-translation-invariant. The autocorrelations $\langle\zeta^+\zeta^+\rangle$ and $\langle\zeta^-\zeta^-\rangle$ are particularly simple and entirely independent of the wavespeed c. Thus the energy spectrum which, for $\lambda_1 = 1$, is

$$E(k) = \frac{1}{2}\int \frac{\mathrm{d}k}{2\pi}\left[\langle |u(k)|^2\rangle + \lambda_2\langle |b(k)|^2\rangle\right] \quad (29)$$

$$= \frac{1+\lambda_2}{2}\int \frac{\mathrm{d}k}{2\pi}\left[\langle |z^+(k)|^2\rangle + \langle |z^-(k)|^2\rangle\right] \quad (30)$$

$$= \frac{1+\lambda_2}{2}\int \frac{\mathrm{d}k}{2\pi}\left[\langle |\zeta^+(k)|^2\rangle + \langle |\zeta^-(k)|^2\rangle\right] \quad (31)$$

is therefore *identical* to that of the randomly forced Burgers equation (25), which does not contain the Alfvén waves at all. This refutes the claim of [10], since the arguments in that work would, if correct, have applied to the present model as well. This establishes the assertion in the introduction that a nonzero mean magnetic field is irrelevant to a determination of the energy spectrum of MHD turbulence. It is worth noting that our result holds regardless of the nature of the forcing (deterministic, stochastic, with or without long-range correlations).

A demonstration that *all* equal-time correlations in (5) are independent of b_0 requires showing that $\langle z^+(k)\, z^-(-k)\rangle$ does not involve the wavespeed. We establish this below for sufficiently small wavenumber. We must assume in our derivation that the long-wavelength behaviour of (25) is determined by a fixed point at which Galilean-invariance prevents the nonlinear coupling in (25) from renormalising, and at which higher-order nonlinearities are not relevant. This is justified by the numerical results of [3] where it is shown that, although higher-order correlations obey multiscaling, the scaling of two-point correlations agrees perfectly with that obtained by arguing that the vertex does not renormalise. In that case, the *renormalised* $\langle z^+(k) z^-(-k)\rangle$ correlation function can be written in terms of the *renormalised* propagator $G(k,t)$ of the randomly-forced Burgers equation (25) and the (renormalised, in principle) correlator $N_{+-}(k,t)$ of $f^+(k,t)$ with $f^-(-k,0)$ with appropriate phase factors arising from the change of variables (18):

$$\langle z^+(k,t) z^-(k',t')\rangle = \mathrm{e}^{-ikct+ik'ct'}\delta(k+k') \times \int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t'}\mathrm{d}t_2 G(k,t-t_1) N_{+-}(k,t_1-t_2) \times \mathrm{e}^{ick(t_1+t_2)} G(-k,t'-t_2) \quad (32)$$

$$\equiv S_{+-}(k,t-t')\delta(k+k'). \quad (33)$$

Redefining $t-t_1 \to t_1$, $t'-t_2 \to t_2$, we obtain the equal-time correlations

$$S_{+-}(k) = \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 G(k,t_1) G(-k,t_2) \times \mathrm{e}^{ikc(t_1+t_2)} N_{+-}(k,t_2-t_1). \quad (34)$$

Scaling tells us that

$$G(k,t) = \Gamma(k^z t) \quad (35)$$

and

$$N_{+-}(k,t) = k^z R_{+-}(k)\Gamma_N(k^z t) \quad (36)$$

where z is the dynamic exponent for equations (25), *i.e.*, without the wave terms, Γ and Γ_N are scaling functions, and $R_{+-}(k)$ is the zero-frequency renormalised covariance of f^+ and f^-. Defining $\tau_1 = k^z t_1$ and $\tau_2 = k^z t_2$, we see that

$$S_{+-}(k) = R_{+-}(k)\int_0^\infty \mathrm{d}\tau_1\int_0^\infty \mathrm{d}\tau_2 \Gamma(\tau_1)\Gamma(\tau_2) \times \Gamma_N(\tau_1-\tau_2)\mathrm{e}^{i(\tau_1+\tau_2)ck^{1-z}}. \quad (37)$$

Since the dynamic exponent governing the decay of correlations in the Burgers equation with noise variance $\sim 1/k$ is known [3,2] to be 2/3, the phase factor inside the integral (37) for small k can be set to unity, *i.e.*, the wavespeed term drops out. This completes the demonstration that equal-time correlations in MHD turbulence are independent of the mean magnetic field[4].

[3] This apparent nonstationarity is of course cancelled by corresponding terms in $\langle\zeta^+\zeta^-\rangle$

[4] Even if we had a model with $z > 1$, say, for a noise with vanishing variance at zero wavenumber, the contribution from (37) would be of the form (noise-strength/ck), while those from the z^+z^+ and z^-z^- would be of order (noise-strength/k^z). The former, while in principle dependent on the wavespeed, would be subdominant (since we are now considering $z > 1$) to the latter.

4 Scaling, renormalisation group and self-consistent analyses for zero mean magnetic field

Some general observations will allow us to estimate the scaling exponents for this 1dMHD model. First the Galileian invariance mentioned before implies that λ_1 will not renormalise. Secondly, the fact that b_0 can be eliminated completely from equation (10) to give equations (25) by the transformations $x \to x \pm ct$ means that $c \equiv \sqrt{\lambda_2} b_0$ cannot renormalise. Thirdly, the invariance of equations (25) under the Galileian transformation $\zeta^{\pm}(x,t) = \zeta^{\pm}(x + \lambda_2 u_0 t, t) \pm u_0$, means that the coupling strength λ_2 in equations (25) cannot renormalise [5]. This automatically implies that the mean magnetic field b_0 also does not renormalise.

Thus, if we were to carry out a renormalisation-group transformation by integrating out a shell of modes $\Lambda e^{-\ell} < q < \Lambda$, and rescaling $x \to e^{\ell} x$, $u \to e^{\ell \chi_u} u$, $b \to e^{\ell \chi_b} b$, $t \to e^{\ell z} t$, the couplings λ_1 and λ_2 would be affected only by the rescaling:

$$\lambda_1 \to e^{\ell(\chi_u + z - 1)} \lambda_1, \lambda_2 \to e^{\ell(2\chi_b - \chi_u + z - 1)} \lambda_2. \tag{38}$$

We can thus rescale to keep λ_1 and λ_2 fixed, giving

$$\chi_u = \chi_b = 1 - z. \tag{39}$$

The noises sources f_u and f_b have variances (see equation (9)) which diverge at small wavenumber k. Since the nonlinear couplings are first order in k, any renormalisation of ϵ_u, ϵ_b and ϵ_{ub} will be of the form $k^2 W(k,\omega)$ which vanishes for $k \to 0, \omega \neq 0$. The noise is thus unrenormalised at small k for any nonzero frequency. If we extend this to say that the noise strength receives no fluctuation corrections at all, then, under a renormalisation-group transformation, the noise strength too is affected only by rescaling. Insisting that the rescaling leave the noise strength unchanged yields $z = 2\chi_u = 2\chi_b$. Thus, $z = 2/3$ and $\chi_u = \chi_b = 1/3$, so that $E(k) \sim k^{-5/3}$. While this treatment, which is the same as that applied to the Burgers equation with noise variance $1/k$, does not yield multiscaling, it appears [3] to be satisfactory for two point correlations. Since our equations for $\zeta^{\pm}$ (25) are identical to [3], it is reasonable to expect that the 1dMHD equations (5) with singular noise will show multiscaling similar to that in [3]. In particular, in this 1d treatment, z^+ and z^- obey the same equation implying that their multiscaling properties are the same. In other words, from the numerical results of [3] we expect $\langle [z^{\pm}(0) - z^{\pm}(x)]^2 \rangle \sim x^{2/3}$ and $\langle [z^{\pm}(0) - z^{\pm}(x)]^p \rangle \sim x$ for $p > 2$. This is consistent with the behaviour of the Elsässer fields in a shell model of 3dMHD [15].

In Section 3 our decoupling of z^+ and z^- required $\mu = \nu$. For $b_0 \neq 0$, we showed that terms involving $\mu - \nu$ were subdominant to the b_0 terms. Since the same arguments cannot be made for $b_0 = 0$, we have carried out a dynamic renormalisation group treatment in the absence of a mean magnetic field, following the momentum shell approach of [16,17]. In our treatment we allow for independent coupling terms $\lambda_1 u \partial_x u$, and $\lambda_2 b \partial_x b$ in the u equation and $\lambda_3 \partial_x (ub)$ in the b equation. We assume, as in the scaling argument above, that there is no diagramatic correction to the noise strength, and we choose the rescaling so that the noise strength remains fixed. For simplicity we ignore the cross-correlation $\langle f_u f_b \rangle$. The resulting recursion relations have a stable fixed point at which $\mu = \nu$. This allows us to work with $\mu = \nu$, and hence to use the decoupled description even when $b_0 = 0$.

We have also carried out a one-loop self-consistent treatment of a slight generalisation of equations (5) and (6) with $A_u(|k|) \sim D_1 k^{-y+3}, A_b(|k|) \sim D_2 k^{-s+3}$ and $kC(|k|) = 0$. Ignoring noise and vertex renormalisations, we find that the fluctuation corrections $\Sigma_u(k)$ and $\Sigma_b(k)$ at zero frequency for ν and μ respectively obey:

$$\Sigma^u(k) = \int \frac{dp}{2\pi} \left\{ \frac{D_1 p^{-y+3}}{p^2 \Sigma^u(p) \left[p^2 \Sigma^u(p) + (k-p)^2 \Sigma^u(k-p) \right]} \right. \tag{40}$$

$$\left. + \frac{D_2 p^{-s+3}}{p^2 \Sigma^b(p) \left[p^2 \Sigma^b(p) + (k-p)^2 \Sigma^b(k-p) \right]} \right\} \tag{41}$$

and

$$\Sigma^b(k) = \int \frac{dp}{2\pi} \left\{ \frac{D_1 p^{-y+3}}{p^2 \Sigma^u(p) \left[p^2 \Sigma^b(p) + (k-p)^2 \Sigma^b(k-p) \right]} \right. \tag{42}$$

$$\left. + \frac{D_2 p^{-s+3}}{p^2 \Sigma^b(p) \left[p^2 \Sigma^u(p) + (k-p)^2 \Sigma^u(k-p) \right]} \right\}. \tag{43}$$

If we seek a solution of the form $\Sigma_u = \Gamma_u k^{x_1}$ and $\Sigma_b = \Gamma_b k^{x_2}$ we find consistency requires $x_1 = x_2 = y/3 = s/3$ and $\Gamma_u = \Gamma_b$, *i.e.*, that the scale-dependent kinematic and magnetic viscosities are identical at small k. The singularities of the one-loop integral depend strongly upon y. For $y \geq 3$, these integrals diverge even when the external wavenumber $k \neq 0$ (the "sweeping divergence" [18,19]). For y near 0, on the other hands, one finds $\Sigma_u(k) \sim \Sigma_b(k) \sim k^{-y/3}$. These singularity structures are very similar to those seen in a self-consistent treatment of the randomly stirred Navier-Stokes equations.

A technical clarification may be of some interest here. In carrying out the perturbation theory with $b_0 = 0$, care must be taken to ensure that the condition $\langle b \rangle = 0$ is maintained order by order. If $ikC(|k|) \neq 0$, *i.e.*, there are cross correlations in the noise sources in equations (5), and $\mu \neq \nu$, then the perturbation theory generates an apparent

[5] This is another point of similarity with 3dMHD: In terms of the Elsässer [14] variables $\mathbf{z}^{\pm}$, the nonlinearity in the 3dMHD equation for $\mathbf{z}^+$ is $\mathbf{z}^- \cdot \nabla \mathbf{z}^+$, and that in the equation for $\mathbf{z}^-$ is $\mathbf{z}^+ \cdot \nabla \mathbf{z}^-$. The equations in this form have the Galileian invariance $\mathbf{z}^{\pm} \to \mathbf{z}^{\pm} + \mathbf{u}_0$, $\mathbf{r} \to \mathbf{r} - \mathbf{u}_0 t$ for any constant vector $\mathbf{u}_0$, where $\mathbf{r}$ is the position coordinate. This guarantees that all the vertices in 3dMHD are unaffected by fluctuation-corrections at zero wavenumber.

non-zero $\langle b \rangle$ and an apparent Alfvén wave speed. Including a counterterm to cancel the spurious $\langle b \rangle$ automatically ensures the absence of Alfvén waves at $b_0 = 0$. At the RG fixed point discussed above, where $\mu = \nu$, this issue clearly does not arise.

Our final point concerns the assertion [11] that the introduction of a nonzero cross-correlation $\langle f_u f_b \rangle$ in 3dMHD generates, under renormalisation, dissipative terms of the form $\nabla^2 \mathbf{B}$ in the equation for $\mathbf{u}$ and $\nabla^2 \mathbf{u}$ in the equation for $\mathbf{B}$. If this were true, the renormalised equations of motion would lack the symmetry properties enjoined on them by the fact that $\mathbf{B}$ and $\mathbf{v}$ are a pseudovector and a vector respectively. Unsurprisingly, a straightforward perturbative analysis of fluctuation corrections arising from the nonlinearities in (5) rules out any such anomaly. As long as the statistics of the noise sources are as in (6a) and (6c), so that the transformation properties of v and b are respected, no such terms are generated. It is clear that the error in [11] arose because their cross-correlation $\langle f_u(k) f_b(-k) \rangle$ was *real* and *even* in wavevector, which is simply not consistent with the nature of the fields in the problem. Such a cross-correlation will in fact generate not only the terms mentioned above but terms like $\partial_x(ub)$ in the u equation and $\partial_x(u^2)$ and $\partial_x(b^2)$ in the b equation as well. The resulting system will have only the invariance $x \to -x, u \to -u$, $b \to -b$, which has nothing to do with the intrinsic transformation properties of the physical fields.

5 Summary

We have shown that a recently proposed [6] Burgers-like one-dimensional model for magnetohydrodynamics (1dMHD) is completely equivalent, through a simple transformation, to existing equations in the literature [7,8]. We have obtained several new results from these 1dMHD equations. The most important of these, which should apply to 3dMHD as well, is that the energy spectrum is unaffected by the presence of a mean magnetic field. Apart from this, we have presented scaling, renormalisation-group and self-consistent analyses of the large-distance, long-time behaviour of correlation functions in the model, and provided arguments for the existence of multiscaling therein when the forcing functions have singular small-wavenumber correlations. Lastly, we have corrected an erroneous claim [11] regarding the effect of cross-correlations between the forcing functions in the velocity and the magnetic field equations.

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Note added in proof

It just came to our attention that the first appearance, in an MHD context, of the 1d model equations used in our paper is: S. Yanase, Phys. Plasmas **4**, 1010 (1997). This predates Fleischer and Diamond (Ref. [6]).

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